REAL OPTIONS AND THE JORGENSONIAN USER COST OF CAPITAL

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Dixit and Pindyck (1994) claim that the Jorgensonian *rule of investment (Jorgenson, 1963) does not hold in uncertain settings. This note shows that the validity of that claim depends crucially on the particular process chosen to describe the economic uncertainty. In fact, the* Jorgensonian *rule of investment holds in other real options models.*

Keywords: Real options, user cost of capital, free-boundary, gamma process.

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1. Introduction

Jorgenson (1963) shows that, under certainty conditions, investing becomes optimal when the marginal return of capital equals the user cost of capital[1]. On the other hand, the real options approach has led to important insights about investment decisions in uncertainty settings, and previous conclusions about investment had to be revised. In particular, Dixit and Pindyck (1994) claim that the neoclassical (*Jorgensonian*) investment rule does not hold in such settings, in which the return of capital must exceed the user cost of capital by a quantity whose magnitude increases with uncertainty. This note shows that the Dixit and Pindyck's claim is not valid in general, but contingent on their particular modeling, according to which a Wiener process describes the economic uncertainty. In the real options framework that we present in Section 3, the *Jorgensonian* investment rule still holds.

[1]The user cost of capital is, in Jorgenson's words, the *shadow* price or implicit rental of one unit of capital. On the other hand, G. Bertola pointed out that the trigger value for the marginal revenue product of capital can be interpreted as the user cost of capital; see Dixit and Pindyck (*op. cit.,* p. 145) and Abel and Eberly (1996, ft. 8).

2. The Dixit-Pindyck framework

Let us recall the assumptions of the basic real options model (McDonald and Siegel, 1986). An investment project demands a sunk cost K, irrespective of when the investment is carried out[2]. The market value of the project, V_t, evolves as:

$$dV_t = \alpha V_t dt + \sigma V_t dW_t, \qquad [1]$$

where α and σ represent drift and volatility constant parameters respectively and W_t is a Wiener process. The investor discounts cashflows at a constant rate $r(> \alpha)$.

Dixit and Pindyck show that it is optimal to invest as soon as the state variable V_t reaches $\frac{\beta}{\beta-1}K$, with $\beta = \frac{1}{2} - \alpha/\sigma^2 + \sqrt{\left(\frac{1}{2} - \alpha/\sigma^2\right)^2 + 2r/\sigma^2}$. As β is larger than one, the NPV-investment rule is suboptimal.

If instead of the previous formulation, we consider that the V_t value comes from a profit flow π_t that satisfies the following dynamics:

$$d\pi_t = \alpha \pi_t dt + \sigma \pi_t dW_t, \qquad [2]$$

the project value V_t is equal to the expected present value of future profits, and it turns out to be $V_t = \pi_t/(r - \alpha)$.[3] The level of profits that makes optimal to invest is $\pi^* = (r - \alpha)\frac{\beta}{\beta-1}K$. This can be written as $\pi^* = \left(r + \frac{1}{2}\sigma^2\beta\right)K$, which is larger than the *Jorgensonian* threshold rK; see Dixit and Pindyck (1994).

3. An alternative real options model

Consider now the same investment problem as before except for in the fact that the project value V_t evolves as follows:

$$dV_t = \alpha V_t dt + V_t X(dt). \qquad [3a]$$

where the source of uncertainty $X(\cdot)$ is the difference of two independent and identically distributed gamma process of mean rate μ and variance rate ν, with $\nu < \mu$:

$$X(dt) \equiv \gamma_1(dt) - \gamma_2(dt). \qquad [3b]$$

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[2] We also assume that capital is not affected by depreciation.

[3] $V_t = \displaystyle\int_t^{+\infty} e^{-r(s-t)} E_t(\pi_s)\, ds = \pi_t/(r - \alpha).$

The Gamma process is explained and used in financial contexts in Heston (1993) and Madan *et al.* (1998). It is a pure-jump process whose increments are strictly positive (so it becomes strictly increasing), with infinite activity. Its probability density function is:

$$f_{\gamma(t)}(x) = \frac{(\mu/\nu)^{\mu^2 t/\nu}}{\Gamma(\mu^2 t/\nu)} x^{\mu^2 t/\nu - 1} e^{-\mu x/\nu}, \quad \text{with } x > 0, \qquad [3c]$$

and its characteristic function is $\varphi_{\gamma(t)}(u) = (1 - iu\nu/\mu)^{-\mu^2 t/\nu}$. All of its moments remain finite, with $E(\gamma(t)) = \mu t$ and $Var(\gamma(t)) = t\nu$.

The difference of two independent and identically distributed gamma processes is also a pure-jump process and can be represented by a Brownian motion without drift evaluated at a gamma random time; it is known as the symmetric Variance Gamma process and represents a generalization of the Wiener process (Madan *et al.*, 1998): as ν tends to zero, $\gamma_1(dt) - \gamma_2(dt)$ approaches a Wiener increment dW. It must be pointed out that we have assumed that γ_1 and γ_2 are gamma processes for the sake of definiteness, but they may be assumed to be any two independent stochastic processes with independent and non-negative increments[4]. This framework allows to take into account that investment-returns uncertainty is often affected by kurtosis effects that Wiener processes can not describe, and if γ_1 and γ_2 are not identically distributed, the model also allows to take into account that uncertainty may be skewed[5].

Denoting the value of the option to invest by $F(V)$ and applying the Bellman equation, we obtain the following ordinary differential equation[6]:

$$rF(V) - \Omega V F'(V) = 0, \qquad [4]$$

which becomes of first-order because pure-jump processes do not involve second-order terms, in contrast to diffusion processes (and in particular that of Wiener). The constant magnitude Ω can be shown to be $\alpha - (\mu^2/\nu) \log(1 - \nu^2/\mu^2)$ (see Appendix). This magnitude is larger than α, and is assumed to be smaller than r.

[4] Alternatively, $X(t)$ may be assumed to be a Brownian motion evaluated at any non-decreasing pure-jump process.

[5] In Boyarchenko's words (2004, p. 558), "option value of investment opportunities has been thoroughly analyzed for *Gaussian* processes. But the real world is not Gaussian". Carr *et al.* (2002) justify that pure-jump processes with an infinite arrival rate of jumps are adequate in describing asset returns uncertainty, and corroborate that claim empirically.

[6] See Appendix.

Solving [4] subject to (a) the free-boundary condition $F(V^*) = V^* - K$, and (b) the smooth-pasting condition $F'(V^*) = 1$, where V^* represents the optimal investment threshold, we obtain:

$$F(V) = (V^* - K)\left(\frac{V}{V^*}\right)^{r/\Omega}, \qquad [5]$$

$$\text{with } V^* = \frac{r}{r-\Omega}K. \qquad [6]$$

Let us now consider that the V_t-value comes from a profit flow π_t satisfying:

$$d\pi_t = \alpha\pi_t dt + \sigma\pi_t X(dt).$$

In this case, the expected present value of future profits turns out to be $V_t = \frac{1}{r-\Omega}\pi_t$.[7]

Therefore, the optimal level of profits that must be reached to carry out the investment project becomes $\pi^* = (r-\Omega)V^* = rK$, the *Jorgensonian* rule of investment. Hence, we can interpret Dixit and Pindyck's deviation from the *Jorgensonian* rule as a consequence of the particular process involved, i.e., the Wiener process, and not as a general investment phenomenon.

4. Conclusions

Many conclusions about investment derived in certainty frameworks must be revised when uncertainty is incorporated to the models. However, some typical conclusions from real options models must be revised as well. One example of this is that more uncertainty always delays investment (Sarkar, 2000). Another example is that under uncertainty, the level of profits that makes it optimal carry out an investment project exceeds the user cost of capital. This note has shown that in general this claim is not true, being its validity dependent on the mathematical specification of the problem. The conclusion becomes valid whenever Wiener processes describe the economic uncertainty, as in Dixit and Pindyck (1994) and the standard real options literature. We provide an alternative real options model where the *Jorgensonian* rule of investment holds.

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[7]In this case, $V_t = \displaystyle\int_t^{+\infty} e^{-r(s-t)} E_t(\pi_s)\, ds = \frac{\pi_t}{r-\Omega}.$

Appendix

The Bellman equation can be written as:

$$rF(V_t)\, dt = E_t\left(dF(V_t)\right).$$

$E_t\left(dF(V_t)\right)$ can be expressed as $F'(V) E_t\left(dV_t\right)$, and:

$$
\begin{aligned}
E_t\left(dV_t\right) &= E\left(V_{t+dt} - V_t \mid V_t\right) = \\
&= E\left(V_t e^{\alpha dt + \gamma_1(dt) - \gamma_2(dt)} - V_t \mid V_t\right) = \\
&= V_t E\left(e^{\alpha dt + \gamma_1(dt) - \gamma_2(dt)} - 1\right) = V_t\left(e^{\Omega dt} - 1\right) = \Omega V_t dt,
\end{aligned}
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with $\Omega = \alpha - \left(\mu^2/\nu\right) \log\left(1 - \nu^2/\mu^2\right)$, so that Equation [4] holds.

(Notice that $E\left(e^{\alpha dt + \gamma_1(dt) - \gamma_2(dt)}\right) = e^{\alpha dt} E\left(e^{\gamma_1(dt) - \gamma_2(dt)}\right)$ $= e^{\alpha dt}\left(1 - \nu^2/\mu^2\right)^{-(\mu^2/\nu)dt}$, given that the characteristic function of X is $\varphi_X(u) \overset{def}{=} E\left(e^{iuX}\right)$ and therefore $E\left(e^X\right) = \varphi_X(-i)$ and $E\left(e^{-X}\right) = \varphi_X(i)$. For a gamma process $\gamma(t)$, the characteristic function is $\varphi_{\gamma(t)}(u) = (1 - iuv/\mu)^{-\mu^2 t/v}$. Then, we obtain $\Omega = \alpha - \left(\mu^2/\nu\right) \log\left(1 - \nu^2/\mu^2\right)$).

The general solution to [4] is $F(V) = BV^{r/\Omega}$, where B is a constant that depends upon the conditions of the problem. In our particular case, the (free) boundary condition is $F(V^*) = V^* - K$. In order to calculate the two unknown magnitudes B and V^* we need a second equation. The smooth-pasting condition is the standard one ($F'(V^*) = 1$, or alternatively, maximize F with respect to V^*). Solving for B and V^* we obtain [5] and [6].

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Resumen

Dixit y Pindyck (1994) establecen que el umbral neoclásico de inversión (Jorgenson, 1963) no es válido en situaciones de incertidumbre. La presente nota demuestra que la validez de dicha conclusión depende del proceso estocástico elegido para describir la incertidumbre subyacente. De hecho, el umbral neoclásico puede resultar válido en otros modelos de opciones reales.

Palabras clave: Opciones reales, procesos gamma, coste de capital Jorgensoniano*.*