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MORE ON THE DYNAMICS IN THE ENDOGENOUS GROWTH MODEL WITH HUMAN CAPITAL

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This paper shows that multiple and globally indeterminate long-run growth rates can easily arise in the two-sector growth model introduced by Lucas (1988). This result is generated by the existence of diminishing returns to time at the private level in human capital accumulation and the existence of external effects from human capital stock in production. The paper asserts that two interior balanced growth paths arise under a sufficiently large elasticity of intertemporal substitution. One is locally determinate, whereas the other can be locally indeterminate. Furthermore, we show that these balanced growth paths can also be globally indeterminate.

Keywords: Endogenous growth, human capital, externalities, multiple equilibria, indeterminacy.

(*JEL* D62, J24, E3, O41)

1. Introduction

This paper obtains additional implications for the equilibrium dynamics in the endogenous growth model introduced by the seminal paper of Lucas (1988). Such a model emphasizes that human capital accumulation is one of the relevant sources of perpetual growth of income. Moreover, the original formulation assumes that the rate of human capital accumulation is linear in time, and considers that the sector producing physical goods exhibits a positive external effect arising from the average stock of human capital. Under these assumptions, the model exhibits dynamics that have received special consideration in the

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literature (see, e.g., Caballé and Santos, 1993, and Mulligan and Sala-i-Martin, 1993). The Lucas model (1988) predicts that growth rates in countries with identical fundamentals tend to converge in the long run, whereas their income levels may be equal only if they have identical departure conditions. Furthermore, Benhabib and Perli (1994) and Xie (1994) prove that the equilibrium can be indeterminate in the sense that there can exist a continuum of equilibrium paths converging to a unique long-run balanced growth path (BGP henceforth). This means that the income levels could be permanently different across countries, even if they have both identical fundamentals and an identical initial composition of physical and human capital.

Therefore, the original Lucas model does not account for multiple long-run growth rates. This result seems partially unsatisfactory. Levine and Renelt (1992) have shown the lack of robustness of the empirical results explaining differences in the growth rate of countries by institutional and policy differences, and by differences in the rate of factor accumulation.[1] Hence, it seems interesting to search for sources of multiplicity in the endogenous growth model with human capital. Ladrón de Guevara, Ortigueira and Santos (1997) obtain multiplicity of BGPs by including pure leisure in the utility function. Their model accounts for a disparity of long-run growth rates depending on the initial endowments of physical capital and human capital. In this case, history then determines the fate of the economy. Chamley (1993) proved that multiple long-run growth rates can appear when an externality arising from the average "learning" time is considered in the sector accumulating human capital. Moreover, Benhabib and Perli (1994) show that if the human capital sector exhibits social increasing returns to time, global indeterminacy is also possible, in the sense that there may exist a range of initial conditions under which there are multiple equilibrium paths leading to different BGPs. In this case, the realization of a specific long-run growth rate depends entirely on the initial decisions of individuals, which are driven by self-fulfilling prophecies. Hence, this model predicts that identical countries can exhibit in the limit not only different income levels, but also different growth rates.

The present paper shows that multiple long-run growth rates are also possible in the Lucas model (1988) without considering either leisure in

[1] Recently, Jones (1995) has obtained further empirical evidence confirming the absence of growth effects of fiscal policies.

the utility function or externalities in the sector accumulating human capital. We extend the original model to allow for a rate of human capital accumulation that is strictly concave in time, together with a human capital externality in production. The assumption of private diminishing returns to "learning" time seems quite appropriate. One of the traditional results obtained by the literature on life-cycle earnings is that the elasticity of human capital accumulation with respect to time is smaller than unity. For instance, in an empirical investigation of the lifetime earnings of the U.S. male high school and college graduates for the period 1960-1970, Rosen (1976) estimated this elasticity to be 0.65. In a general equilibrium framework, Lucas (1990a) estimated a value of 0.8 for such elasticity using U.S. data from 1955 to 1985.[2]

In the previous environment, two BGPs emerge when the externality and the elasticity of intertemporal substitution are both sufficiently large. One of them is locally determinate, whereas the other is either locally indeterminate or locally unstable. Moreover, given this local dynamic behavior, the multiplicity of BGPs may also render global indeterminacy of equilibrium. Hence, multiplicity and global indeterminacy are also feasible in a simpler version of the Lucas model (1988), so that these phenomena are more pervasive than one could expect.

The mechanism generating multiplicity and global indeterminacy in our model is quite straightforward. There are two countervailing forces affecting the allocation of time to human capital accumulation. On the one hand, the presence of a human capital externality generates a complementarity between the accumulation of human capital and the production of physical goods. Thus, the time currently spent in accumulating human capital will increase the marginal productivity of human capital in the sector producing physical goods, and so the return to human capital investments. This fact stimulates individuals to reallocate time to human capital accumulation and to substitute future consumption for present consumption. On the other hand, diminishing returns to time in human capital accumulation discourages individuals from spending a large amount of time in order to obtain an additional unit of new human capital, which raises present consumption. If the elasticity of intertemporal substitution is sufficiently high,

[2]This assumption has already been assumed in different versions of the two-sector model of endogenous growth (see, e.g., Uzawa, 1965, Lucas, 1990a, and Caballé and Santos, 1993). However, none of these models allowed for the existence of an externality in production.

any of these two forces can dominate, so that individuals' expectations determine the allocation of time between the two sectors. For instance if individuals expect a high return to human capital investments, they allocate a larger fraction of time to accumulating human capital in spite of the diminishing returns. Because of the externality, the expectations become self-fulfilling. The result is a model in which there may be two BGPs, each of which is defined by an alternative fraction of time devoted to accumulating human capital.

Before closing this section, we would like to give some insights on the empirical relevance of our result. First, this paper shares with the previous endogenous growth models with human capital the result that equilibrium indeterminacy needs an unrealistically high elasticity of intertemporal substitution. However, this problem may be solved by incorporating other assumptions in the model. We will not incorporate them to avoid their hiding the mechanisms generating the multiplicity and indeterminacy of equilibria. In any case, in the conclusion of the paper we will mention a few mechanisms which would serve as a step in this research. Secondly, indeterminacy implies a problem of coordination failure. How do individuals coordinate their expectations for selecting an equilibrium path? Evidently, when multiple equilibrium paths exist, they can be ranked following the welfare criterion. However, the question is how individuals agree to select the equilibrium path generating the highest welfare. Hence, one should further look for alternative mechanisms of equilibrium selection. Nevertheless, this question remains open since this is not the scope of the paper.

The rest of the paper is organized as follows. The model is presented in Section 2, where the competitive equilibrium of the economy is characterized. Section 3 discusses the necessary and sufficient conditions for the existence of multiple BGPs, whereas Section 4 analyzes the global equilibrium dynamics of the model. Section 5 closes the paper with some concluding remarks.

2. The model

We extend the two-sector model of endogenous growth introduced by Lucas (1988) to allow for a strictly concave rate of human capital accumulation. Formally, as in Lucas (1990a), the technology of human capital accumulation we postulate is $\dot{h}(t) = h(t)\gamma(1-u(t))^{1-\alpha}$, where $h(t)$ is human capital, $1-u(t)$ is the fraction of non-leisure time that each individual allocates to accumulating human capital through nonmarket activities, γ is a positive technology parameter, and α is a constant belonging to the open interval (0,1).

We now turn to the exposition and analysis of the model. The problem each individual faces in a decentralized economy is given by

$$\max_{\{C(t),u(t)\}} \int_0^\infty \left[\frac{C(t)^{1-\sigma}-1}{1-\sigma}\right] e^{-\rho t} dt, \qquad [P]$$

subject to

$$\begin{aligned} \dot{K}(t) &= AK(t)^\beta (u(t)h(t))^{1-\beta} h_a(t)^v - C(t), & [1] \\ \dot{h}(t) &= h(t)\gamma[1-u(t)]^{1-\alpha}, & [2] \\ C(t) &\geq 0, \;\; u(t) \in [0,1], \;\; K(t) \geq 0, \;\; h(t) \geq 0, & \\ K(0) &= K_0, \;\; h(0) = h_0, & \end{aligned}$$

where $C(t)$ is consumption, $K(t)$ is physical capital, $h_a(t)$ is the average level of human capital, $u(t)$ is the fraction of time allocated to the production of physical goods, A is a positive technology parameter, β is the share of physical capital, ν is a positive externality parameter in the accumulation of human capital, ρ is a positive discount rate, and $\sigma > 0$ is the inverse of the intertemporal elasticity of substitution. For notational convenience, we index the fundamentals of the model by the vector of parameters θ. We then define $\theta \equiv (A, \gamma, \rho, \sigma, \nu, \alpha, \beta)$, and $\theta \in \Theta$, where $\Theta = R^4_{++} \times R_+ \times (0,1)^2$.

The problem [P] is a standard dynamic optimization problem with control variables $C(t)$ and $u(t)$, and state variables $K(t)$ and $h(t)$. Notice that individuals take $h_a(t)^v$ as an exogenously given function of time. The necessary conditions for an optimum can be found from the solution of the current-value Hamiltonian equation[3]

$$\begin{aligned} H(C,u,K,h,\lambda_1,\lambda_2) = \frac{C^{1-\sigma}-1}{1-\sigma} + \\ \lambda_1(AK^\beta(uh)^{1-\beta}h_a^v - C) + \lambda_2(h\gamma(1-u)^{1-\alpha}), \end{aligned} \qquad [3]$$

where λ_1 and λ_2 are the associated costate variables. By the standard procedure, we find the first order conditions, then substitute in the consistency equilibrium condition $h = h_a$, and finally rearrange the expressions to eliminate λ_1 and λ_2. The necessary conditions for

[3]Except when necessary, we will suppress hereafter the time argument of all endogenous variables so as to ease the notation.

optimality are summarized by an autonomous system of differential equations in (K, h, C, u):

$$\dot{K} = AK^{\beta}u^{1-\beta}h^{1-\beta+v} - C, \quad [4]$$

$$\dot{h}(t) = h(t)\gamma\left[1 - u(t)\right]^{1-\alpha}, \quad [5]$$

$$\frac{\dot{C}}{C} = \frac{1}{\sigma}\left[\beta AK^{\beta-1}u^{1-\beta}h^{1-\beta+v} - \rho\right], \quad [6]$$

$$\dot{u} = \frac{u(1-u)}{\alpha u + \beta(1-u)} \times \left\{\gamma(1-\alpha u)(1-u)^{-\alpha} - \gamma(\beta-\nu)(1-u)^{1-\alpha} - \beta\frac{C}{K}\right\}. \quad [7]$$

Since the associated Hamiltonian is jointly concave in state and control variables, the first order conditions are also sufficient if in addition the following transversality conditions are fulfilled:

$$\lim_{t\to\infty} e^{-\rho t}\lambda_1 K = 0, \quad [8]$$

$$\lim_{t\to\infty} e^{-\rho t}\lambda_2 h = 0. \quad [9]$$

DEFINITION 1. A competitive equilibrium consists of paths $\{K(t), h(t), C(t), u(t)\}$ that satisfy equations [4] to [9] for the given initial conditions K_0 and h_0.

DEFINITION 2. A BGP (or steady-state equilibrium) is a set of paths $\{K(t), h(t), C(t), u(t)\}$ satisfying Definition 1, such that the variables $K(t), h(t)$, and $C(t)$ grow at a constant rate, and $u(t)$ is constant.

Since the actual data show a positive investment in physical and human capital, we shall only focus on cases in which the economy exhibits interior BGPs, i.e., $0 < u(t) = u^* < 1$. The next sections will discuss the existence of interior BGPs and the equilibrium dynamics. We will use both analytical and numerical arguments for these purposes.

3. Existence and properties of the steady-state equilibria

In this section we investigate the properties of the interior BGP and the conditions for its existence. First, we determine the relationship among the growth rates of variables along any BGP. We obtain the same result as Lucas (1988, p. 22).

PROPOSITION 1. *Let g_k^*, g_h^* and g_c^* denote respectively the growth rates of K, h and C in a BGP. Then, $g_k^* = g_c^* = ((1-\beta+\nu)/(1-\beta))g_h^*$.*

PROOF. First, combining [6] with [4], we prove that K and C must grow at the same rate on a BGP. Given this, and since u is constant along a BGP, we prove directly from [4] that $g_h^* = ((1-\beta)/(1-\beta+\nu))g_k^*$. Q.E.D

In order to proceed with our analysis of the interior BGPs and the equilibrium dynamics, we introduce the next two transformed variables:

$$z = Kh^{\frac{1-\beta+\nu}{\beta-1}}, \qquad [10]$$

$$x = \frac{C}{K} \qquad [11]$$

These new variables remain constant along a BGP. By taking logarithms on both [10] and [11], and differentiating with respect to time, we derive the dynamic equations for z and x. Thus, the dynamic system [4]-[7] can be reduced to the following three dimensional system in (z, x, u):

$$\dot{z} = z\{Az^{\beta-1}u^{1-\beta} - x - \psi(1-u)^{1-\alpha}\}, \qquad [12]$$

$$\dot{x} = x\{\frac{\beta-\sigma}{\sigma}Az^{\beta-1}u^{1-\beta} + x - \frac{\rho}{\sigma}\}, \qquad [13]$$

$$\dot{u} = \frac{u(1-u)}{\alpha u + \beta(1-u)}\{\gamma(1-\alpha u)(1-u)^{-\alpha} - \gamma(\beta-\nu)(1-u)^{1-\alpha} - \beta x\}, \qquad [14]$$

where $\psi = (\gamma(1-\beta+\nu))/(1-\beta)$. Since the initial value of z is fully defined by K_0 and h_0, given these initial conditions, the dynamic equations [12]-[14] plus the transversality conditions [8] and [9] summarize the competitive equilibrium paths.[4] Thus, the stationary points of this reduced system are the candidates to define the interior BGPs. We next compute the interior steady states of the reduced system [12]-[14].

LEMMA 1. *Let z^*, x^* and u^* be the stationary values of z, x and u, respectively. The interior steady states of the dynamic system [12]-[14] are the solutions of the following system of ordinary equations:*

$$x^* = \frac{\gamma(1-\alpha u^*)}{\beta}(1-u^*)^{-\alpha} - \frac{\gamma(\beta-\nu)}{\beta}(1-u^*)^{1-\alpha}, \qquad [15]$$

[4]Note that the new dynamic system [12]-[14] has z as a state-like variable, and x and u as control-like variables.

$$z^* = u^* \left[\frac{\beta\rho + \sigma\gamma(\beta-\nu)(1-u^*)^{1-\alpha} - \sigma\gamma(1-\alpha u^*)(1-u^*)^{-\alpha}}{A\beta(\beta-\sigma)} \right]^{1/(\beta-1)}, \quad [16]$$

$$\gamma\left[\nu - \sigma(1-\beta+\nu)\right](1-u^*)^{1-\alpha} + \gamma(1-\beta)(1-\alpha u^*)(1-u^*)^{-\alpha} = \rho(1-\beta). \quad [17]$$

Proof.The results directly follow from setting the system [12]-[14] expressed in terms of growth rates equal to zero. Q.E.D.

The steady states of the reduced system correspond to interior BGPs if they satisfy the transversality conditions [8] and [9]. Note that the latter two conditions are satisfied if the following inequalities hold:

$$\lim_{t\to\infty}\left(-\rho + \frac{\dot{\lambda}_1}{\lambda_1} + \frac{\dot{K}}{K}\right) < 0,$$

$$\lim_{t\to\infty}\left(-\rho + \frac{\dot{\lambda}_2}{\lambda_2} + \frac{\dot{h}}{h}\right) < 0.$$

Substituting for the growth rate of both the state and costate variables, we get that the previous inequalities evaluated at the steady states given by the system [15]-[17] reduce to $-\gamma(1-\alpha)u^*(1-u^*)^{-\alpha} < 0$, which always holds for u^* belonging to the open interval (0,1). Therefore, we must impose necessary and sufficient conditions for the existence of an interior BGP for which $u^* \in (0,1)$.

Proposition 2: *Consider the following subsets of* Θ*:*

$$\begin{aligned}
\Theta_1 &\equiv \{\theta\in\Theta \mid \sigma > 1-\rho/\psi\},\\
\Theta_2 &\equiv \{\theta\in\Theta \mid \sigma = 1-\rho/\psi \text{ and } \nu-\sigma(1-\beta+\nu)>0\},\\
\Theta_3 &\equiv \{\theta\in\Theta \mid \sigma < 1-\rho/\psi,\ \nu-\sigma(1-\beta+\nu)>0\\
&\qquad \text{and } \gamma\eta^\alpha = \rho < \psi\},\\
\Theta_4 &\equiv \{\theta\in\Theta \mid \sigma < 1-\rho/\psi,\ \nu-\sigma(1-\beta+\nu)>0\\
&\qquad \text{and } \gamma\eta^\alpha < \rho < \psi\},\\
\Theta_5 &\equiv \Theta\setminus \cup_{i=1}^4 \Theta_i,
\end{aligned}$$

where $\eta = 1 + \left[(\nu-\sigma(1-\beta+\nu))/(\alpha(1-\beta))\right]$.

Then, (i) if $\theta \in \cup_{i=1}^3\Theta_i$*, there exists a unique interior BGP, which is associated with the triple* (z^*, x^*, u^*)*. (ii) If* Θ_4*, there exist two interior BGPs, which are respectively associated with the triples* (z_1^*, x_1^*, u_1^*) *and* (z_2^*, x_2^*, u_2^*)*, with* $u_1^* > u_2^*$*. (iii) No interior BGP exists if* $\theta \in \Theta_5$*.*

Proof: See Appendix A1.

REMARK 1: The subspaces of parameters Θ_2 and Θ_3 are exclusively composed by isolated points in Θ. Hence, the unique interior BGP that exists when θ belongs to $\Theta_2 \cup \Theta_3$ is a non-generic equilibrium since an arbitrarily small perturbation of some parameter may alter its properties. More precisely, if θ initially lies in $\Theta_2 \cup \Theta_3$, a marginal change in parameters can move θ to Θ_1, Θ_4 or Θ_5, so that the existence, uniqueness and stability properties of the BGP could change.[5] Consequently, from now on we will ignore the study of these non-generic cases.

Proposition 2 shows that uniqueness of interior BGPs is not guaranteed. There may exist two long-run growth rates. It is easy to provide an example of an economy for which there exists a unique interior BGP. Following Lucas (1988, 1990a, and 1990b) and Mulligan and Sala-i-Martín (1993), we consider the following set of structural parameters: $A = 1, \beta = 0.25, \nu = 0.36, \alpha = 0.2, \gamma = 0.1, \rho = 0.065$ and $\sigma = 2$. It is easy to check that this particular vector of parameters lies in Θ_1, and that the unique interior BGP is defined by $u^* = 0.845382$. The first row of Table 1 presents the stationary values of z, x and u, as well as the growth rate of *per capita* income g^*, the growth rate of human capital g_h^*, the rental rate of physical capital R^*, and the saving rate s^* for the previous numerical model.[6]

An example of the existence of two interior BGPs can also be found easily. First, the elasticity of intertemporal substitution must be sufficiently high. Moreover, in this case the scale factor γ must be sufficiently small to ensure the existence of generic, interior BGPs. Hence, we can consider the same set of parameters as in the previous example, but changing σ to 0.15 and γ to 0.055. One can check that this particular vector of parameters belongs to Θ_4, and the two interior BGPs are defined by $u_1^* = 0.617063$ and $u_2^* = 0.503703$. The second row of Table 1 presents the stationary values of the relevant variables

[5] A particular case of interest could be when θ goes from Θ_4 to Θ_5 through an element in Θ_3. This occurs when the first inequality of the third condition defining Θ_4 is altered *ceteris paribus* by a continuous change in a parameter. In this case, two interior BGPs merge into a single one at Θ_3, and then they disappear. Furthermore, from the stability properties given in Section 4 we would derive that the dynamic system undergoes a saddle-node bifurcation at this vector θ in Θ_3.

[6] We take a broad definition of gross saving. As is usual, we assume that saving is the proportion of output that is not consumed. However, following Mulligan and Sala-i-Martin (1993), we take an extensive measure of output by adding the production of human capital, multiplied by the shadow price of human capital in units of physical goods. Hence, $s(t) = 1 - (C(t)/Q(t))$, where $Q(t) = Y(t) + (\lambda_2(t)/\lambda_1(t))\gamma(1 - u(t))^{1-\alpha}h(t)$.

for the economy represented by this particular model.

TABLE 1
Steady-state equilibria for alternative parameter sets

	Model 1[(a)]	Model 2[(b)]
u^*	0.845	u^*_1=0.617 and u^*_2=0.504
x^*	0.493	x^*_1=0.245 and x^*_2=0.241
z^*	1.991	z^*_1=3.326 and z^*_2=2.650
g^*	0.033	g^*_1=0.038 and g^*_2=0.046
g^*_h	0.022	g^*_{h1}=0.026 and g^*_{h2}=0.031
R^*	0.117	R^*_1=0.071 and R^*_2=0.072
s^*	0.200	s^*_1=0.452 and s^*_2=0.564

Note· *(a) A=1, β=0.25, ν=0.36, α=0 2, γ=0.1, ρ=0 065, σ=2.*
(b) A=1, β=0 25, ν=0.36, α=0.2, γ=0 055, ρ=0 065, σ=0.15.

We observe that the strict concavity of human capital accumulation and the externality in production are necessary conditions for the existence of two interior BGPs. Note that the vector of parameters θ never belongs to the subset Θ_4 when $\nu = 0$, whereas if $\alpha = 0$ the right hand side of [17] is always monotone in u, so that at most one interior BGP exists.[7] However, once we have assumed that $\alpha \in (0,1)$ and $\nu > 0$, the inverse of the elasticity of intertemporal substitution is the crucial parameter that determines the number of interior BGPs, provided that the parameter γ has been controlled to ensure the existence of interior BGPs. Thus, when σ is sufficiently large (small) the economy has one (two) interior BGP(s). For instance, in our second numerical example, $\sigma = 0.201474$ is the bifurcation value of the inverse of the elasticity of intertemporal substitution at which the number of interior BGPs changes.

4. Equilibrium dynamics

We will now describe the behavior of the system outside of the steady-state equilibria. We will investigate whether or not the economy converges to the steady-state equilibria, and whether equilibria are either determinate or indeterminate. We will see that the equilibrium dynamics in the case of a unique BGP are qualitatively identical to those of the Lucas model. However, we are especially interested in describing the equilibrium dynamics when two BGPs exist. Before we attempt a

[7] Caballé and Santos (1993) prove that without externalities and with strictly concave human capital accumulation there always exists a unique interior BGP. On the other hand, Benhabib and Perli (1994) prove the uniqueness of an interior BGP under linearity of human capital accumulation and an externality in production.

characterization of the global dynamics, we must start by examining the local behavior of equilibrium paths around each BGP.

4.1 Local stability properties of the interior BGPs

The local dynamics of the system around a steady state are determined by the signs of the eigenvalues of the Jacobian matrix corresponding to the linearized system. The next result establishes the local stability of the interior BGPs.

PROPOSITION 3: *Consider the economy in Proposition 2. Hence,*

(i) If $\theta \in \Theta_1$, then the unique interior BGP is locally a saddle path, i. e., the equilibrium is locally determinate.

(ii) If $\theta \in \Theta_4$, then the equilibrium is locally determinate around the interior BGP given by (z_1^*, x_1^*, u_1^*), whereas the equilibrium is either locally indeterminate or locally unstable around the interior BGP defined by (z_2^*, x_2^*, u_2^*). A sufficient condition for the equilibrium being locally indeterminate around (z_2^*, x_2^*, u_2^*) is $\alpha u_2^* + (\beta - v)(1 - u_2^*) < 0$.

PROOF: see Appendix A2.

We now provide some numerical examples to illustrate the previous stability properties. Table 2 presents the eigenvalue structure of the Jacobian matrix at each of the steady-state equilibria. Model 1 corresponds to a example where θ belongs to Θ_1, whereas Models 2, 3 and 4 correspond to examples where θ belongs to Θ_4. Moreover, the last three examples allow us to derive some additional conclusions about the local stability of (z_2^*, x_2^*, u_2^*). First, complex conjugate eigenvalues exist for some regions of Θ_4. Hence, the dynamic system [12]-[14] can exhibit transitional oscillations around (z_2^*, x_2^*, u_2^*).

Second, we observe that variations in the parameter values can generate changes in the eigenvalue structure of the Jacobian matrix evaluated at (z_2^*, x_2^*, u_2^*). For instance, when the vector of parameters goes from Model 3 to Model 4, the two negative eigenvalues of the Jacobian matrix transform into positive. In general, we can then divide Θ_4 in two regions: one where the BGP given by (z_2^*, x_2^*, u_2^*) is locally indeterminate and the other where this BGP is locally unstable. At the boundary between both regions, which will generically be denoted by $\bar{\theta}$, the variation in the eigenvalue structure occurs when the real parts of two complex conjugate eigenvalues change sign since the determinant of the Jacobian matrix does not become zero at $\theta = \bar{\theta}$.[8] Hence, at $\theta = \bar{\theta}$ the Jacobian matrix has two pure imaginary eigenvalues together with the positive one, so that the dynamic system undergoes a Hopf bifurcation.[9] Taking a sufficiently small neighborhood of $\bar{\theta}$, an invariant closed curve then emerges around the BGP associated with (z_2^*, x_2^*, u_2^*) for those vectors of parameters on one side of $\bar{\theta}$. As θ changes from one side of $\bar{\theta}$ to the other, the closed curve shrinks and collapses into the BGP at $\theta = \bar{\theta}$. Furthermore, these closed curves will be stable if they appear in the region of Θ_4 where the BGP is locally unstable, whereas they will be unstable if they emerge in the region where the BGP is locally indeterminate. However, in this paper we do not investigate which of these two cases occurs since this requires complicated analytical arguments, and moreover we do not need it to show the global indeterminacy result.

TABLE 2
Eigenvalue structure for alternative equilibria

		μ_1	μ_2	μ_3
Model 1[(a)]	(z^*, x^*, u^*)	-0.208995	0.264188	0.114964
Model 2[(b)]	(z_1^*, x_1^*, u_1^*)	-0.247278	0.287887	0.004491
	(z_2^*, x_2^*, u_2^*)	-0.242879	0.277599	-0.00340
Model 3[(c)]	(z_1^*, x_1^*, u_1^*)	-0.022383	0.039092	0.085295
	(z_2^*, x_2^*, u_2^*)	-0.007+0.004i	-0.007-0.004i	0.033725
Model 4[(d)]	(z_1^*, x_1^*, u_1^*)	-0.013284	0.033821	0.079631
	(z_2^*, x_2^*, u_2^*)	0.0025+0.013i	0.0025-0.013i	0.060876

Note (a) $A=1$, $\beta=0.25$, $v=0.36$, $\alpha=0.2$, $\gamma=0.1$, $\rho=0\ 065$, $\sigma=2$
(b) $A=1$, $\beta=0.25$, $v=0.36$, $\alpha=0.2$, $\gamma=0.055$, $\rho=0.065$, $\sigma=0.15$.
(c) $A=1$, $\beta=0.8$, $v=0.1$, $\alpha=0\ 04$, $\gamma=0\ 05$, $\rho=0.055$, $\sigma=0.25$
(d) $A=1$, $\beta=0.85$, $v=0.1$, $\alpha=0.04$, $\gamma=0.05$, $\rho=0.055$, $\sigma=0\ 25$.

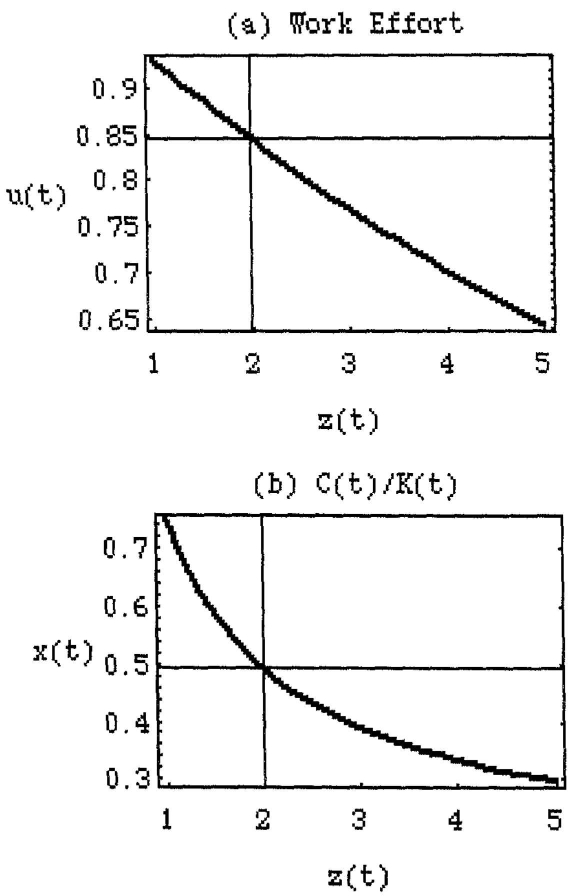
4.2 Global equilibrium dynamics

Knowing the precise behavior of the economic system outside of steady-state equilibria requires a complete characterization of the global equilibrium dynamics. For this purpose, we use the time-elimination method introduced by Mulligan (1991).[10] In this way, we can numerically represent the graphs of the policy functions that relate x and u to z. Furthermore, from these policy functions we could characterize the equilibrium paths of any other relevant variable.

[8] Lemma 2 in Appendix A2 proves that this determinant remains negative when the change in the eigenvalue structure takes place.
[9] Taking the Models 3 and 4 of Table 3, a numerical example with pure imaginary eigenvalues appears by continuity for a particular value of β between 0.8 and 0.85.
[10] See also Mulligan and Sala-i-Martin (1993).

We first describe the global dynamics for an economy with a unique BGP. Thus, Figure 1 displays the equilibrium dynamics for the economy defined by Model 1 of Table 3. In this case, the two policy functions $x(t)$ and $u(t)$ are both composed of a single curve that is downward sloping. These transitional dynamics are then qualitatively identical to those characterized by Mulligan and Sala-i-Martin (1993) for the normal growth case ($\beta < \sigma$) in the Uzawa-Lucas model without externalities.[11]

FIGURE 1
The stable path of the BGP given by (z^*, x^*, u^*)



Global dynamics are quite different when the economy exhibits two interior BGPs. For instance, consider the economy represented by Model 2 of Table 3. In this case, the policy functions $x(t)$ and $u(t)$ are both composed of a multiple paths. One of them goes through the BGP given by (z_1^*, x_1^*, u_1^*), whereas all the others cross the BGP

[11] See Caballé and Santos (1993) for more details.

defined by (z_2^*, x_2^*, u_2^*). Figure 2 displays the projections of the unique saddle path crossing (z_1^*, x_1^*, u_1^*). The graphs of $x(t)$ and $u(t)$ are now

FIGURE 2
The stable path of the BGP given by (z_1^*, x_1^*, u_1^*)

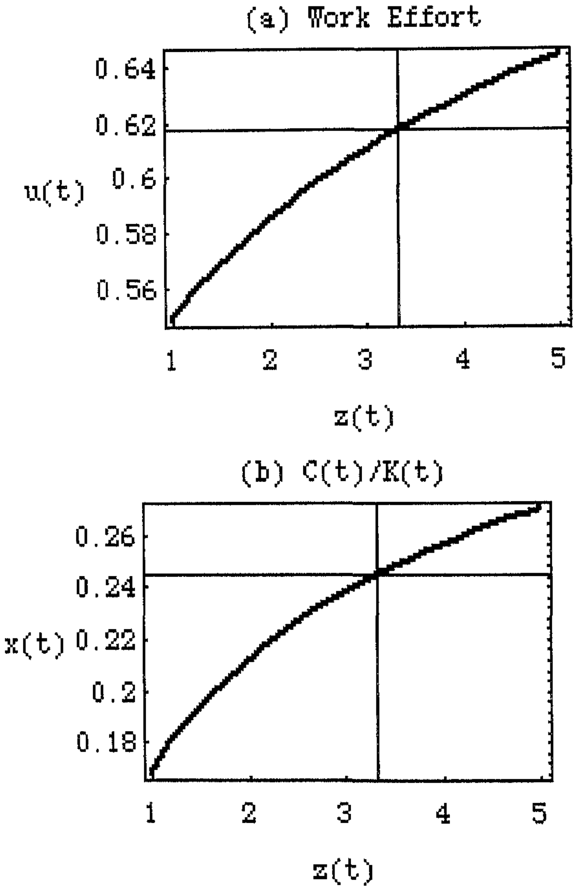
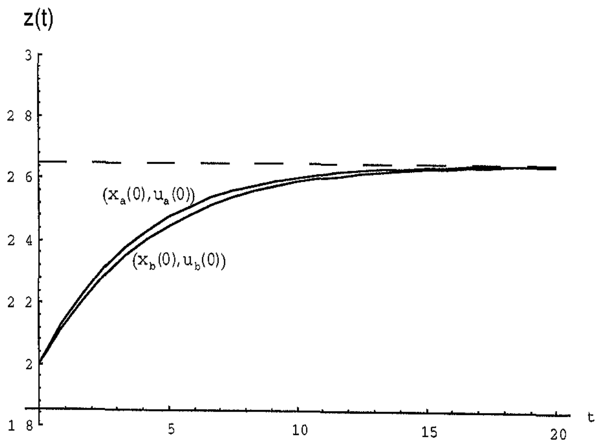


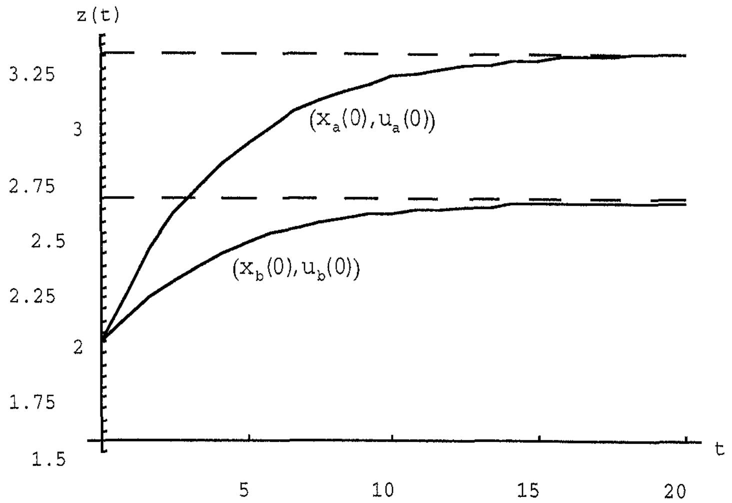
FIGURE 3
Local indeterminacy of the BGP given by (z_2^*, x_2^*, u_2^*)



upward sloping as in the paradoxical growth case ($\beta > \sigma$) of the Uzawa-Lucas model without externalities. In Figure 3, we instead illustrate graphically that the BGP given by (z_2^*, x_2^*, u_2^*) is locally indeterminate. We depict two equilibrium paths for $z(t)$, which are built from the same initial condition $z(0) = 2$, but different starting values for x and u, namely $(x_a(0), u_a(0)) = (0.223785, 0.484853)$ and $(x_b(0), u_b(0)) = (0.233166, 0.517079)$. It can be observed that the initial choices of x and u determine the growth rate of *per capita* income during the transition to the BGP given by (z_2^*, x_2^*, u_2^*), where this growth rate will be constant and equal to 0.046475.

Moreover, putting together the path converging to (z_1^*, x_1^*, u_1^*) and the ones converging to (z_2^*, x_2^*, u_2^*), we observe that Model 2 also exhibits global indeterminacy in the sense that the initial choices of x and u crucially determine the long-run growth rate. Figure 4 draws two equilibrium paths for $z(t)$, which exhibit the same initial condition $z(0) = 2$, but different starting values for x and u, namely $(x_a(0), u_a(0)) = (0.212332, 0.585321)$ and $(x_b(0), u_b(0)) = (0.223812, 0.484831)$. In this case, the initial choices of x and u determine not only the growth rate during the transition, but also the long-run growth rate. Thus, if the initial choice is $(x_a(0), u_a(0))$ the stationary growth rate will be 0.037768, whereas such a rate will be 0.046475 when the initial choice is $(x_b(0), u_b(0))$.

FIGURE 4
Global indeterminacy



REMARK 2: We can summarize the results of the numerical analysis given above to state a conclusion concerning the global uniqueness of the dynamic equilibrium. When θ belongs to Θ_1, the equilibrium is glo-

bally unique. However, when θ lies in Θ_4, we distinguish two different situations. If the BGP associated with (z_2^*, x_2^*, u_2^*) is locally unstable and there are no limit cycles surrounding it, then the equilibrium is globally unique and it converges to the BGP given by (z_1^*, x_1^*, u_1^*). Otherwise, we conjecture that the equilibrium is globally indeterminate at least for z_0 close to z_2^*. In this case, there is a range of initial conditions for z_0 in which the equilibrium trajectory is indeterminate, and may converge to either of the BGPs depending on the initial choice of x and u.

5. Concluding remarks

This paper has extended the endogenous growth model introduced by Lucas (1988) to allow only for a rate of human capital accumulation that is strictly concave in time. We have proved that the following three requirements must jointly be satisfied for the existence of two BGPs and for the equilibrium dynamics described in this paper: the accumulation of human capital must be strictly concave in time, an average human capital externality must exist in production, and the elasticity of intertemporal substitution must be sufficiently large. Otherwise, the dynamic system defining the equilibrium paths exhibits a unique BGP. However, the stability properties of this unique BGP will be different depending on which of the last conditions fails. When the strict concavity of human capital accumulation does not hold, the new BGP is either locally indeterminate or locally unstable. On the other hand, the new BGP is locally determinate when one of the other two conditions fails. The latter case also occurs when any two of the three conditions are jointly violated.

We can then conclude that some properties of the equilibrium dynamics pointed out by Benhabib and Perli (1994) for the original Lucas model (1988) are not generic. These results are based on the assumption of a linear rate of human capital accumulation, so that a marginal variation in the degree of the concavity of this rate can drastically alter the results as we have shown in this paper. Together with the multiplicity of long-run growth rates, we observe that our version reveals other differences with respect to the Lucas model. First, it is easy to check that the subspace of parameters for which a unique and locally determinate BGP exists is larger in our model. Unlike the original Lucas model, the previous case occurs in our model without imposing the discount rate of preferences to be smaller than the scale parameter of the technology for human capital accumulation. The trade-off between "thriftiness" and growth pointed out by Lucas (1988, p.23) is solved in our model in favor of the latter since there are decreasing returns to human capital investment. On the other hand, the range of parameters for which indeterminacy appears is smaller in our model, though we can also obtain global indeterminacy. In our model the condition forcing the discount rate of preferences to be larger than the scale factor of the technology for human capital accumulation does not ensure the existence of at least one interior BGP, provided the elasticity of intertemporal substitution is sufficiently large.[12]

Obviously, we must also notice at this point that the requirement of a large elasticity of intertemporal substitution is an obvious limitation for the empirical relevance of our multiplicity result. However, one could solve this problem by introducing in our model either (qualified) leisure in the utility function as Ladrón de Guevara, Ortigueira and Santos (1997) or home production as Benhabib, Rogerson and Wright (1991). In this way, individuals can reallocate time to the sector accumulating human capital without reducing the time devoted to producing consumption goods. Hence, no restriction on the elasticity of intertemporal substitution would be necessary to obtain the multiplicity result. Since the aim of the paper is to show that diminishing returns in the accumulation of human capital is a sufficient condition for the existence of multiple BGPs and global indeterminacy of the equilibrium, we have omitted these solutions in order to make this conclusion more transparent. Future research should consider this issue as a way of studing the empirical consequences of the source of indeterminacy presented in this paper.

Appendix A1. Proof of Proposition 2

From [15] and [16] we observe that z^* and x^* are continuous functions of u^*. Hence, for the purpose of this proof, we must only determine the conditions under which at least one interior solution of [17] exists. Define from equation [17] the following function of u:

$$\begin{aligned} P(u) \; = \; & [\nu - \sigma(1-\beta+\nu)]\,\gamma(1-u)^{1-\alpha} + \\ & (1-\beta)\gamma(1-\alpha u)(1-u)^{\alpha} - (1-\beta)\rho, \end{aligned} \quad \text{[A1.1]}$$

which has domain $[0,1]$. This function satisfies the following properties:

[12] Notice that the first two conditions defining the subspace of parameters for which multiple BGPs arise jointly imply this requirement. However, the existence of interior BGPs also requires a third condition.

(i) It is twice continuously differentiable in $(0,1)$, with

$$P'(u) \equiv \frac{dP(u)}{du} = \frac{\gamma(1-\alpha)}{(1-u)^{\alpha}}\left\{\frac{\alpha(1-\beta)u}{(1-u)} - [\nu - \sigma(1-\beta+\nu)]\right\}. \quad \text{[A1.2]}$$

(ii) $P(0) \gtreqless 0$ if and only if $\sigma \lesseqgtr 1-(\rho/\psi)$.

(iii) $P(u)$ converges to infinity when u approaches one. Hence, there exists some number ε larger than zero, such that $P(u)$ is strictly positive for all u belonging to $(1-\varepsilon, 1)$.

We next search for the interior solutions of $P(u)=0$. Let us distinguish the following cases.

Case A. Assume $P(0)>0$. First we must impose a technical condition that guarantees that the parameter space defined by the condition $\sigma < 1-(\rho/\psi)$ is not empty. Since we must have $\sigma>0$, the required condition is $\rho<\psi$. We can already characterize the interior solutions under the present situation. If $\nu-\sigma(1-\beta+\nu)\leq 0$, then $P(u)$ is an increasing function, and so there does not exist any u^* for which $P(u^*)=0$. On the other hand, if $\nu-\sigma(1-\beta+\nu)>0$, the derivative $[A1.2]$ is negative for values of u close to zero, whereas such a derivative is positive for values of u close to one. Moreover, $P(u)$ is a quasiconvex function in $(0,1)$. Hence, there exists a unique value of u belonging to $(0,1)$, say u^c, which equals such a derivative to zero. From $[A1.2]$ we obtain this unique critical value of u as

$$u^c = \frac{\nu-\sigma(1-\beta+\nu)}{\alpha(1-\beta)+\nu-\sigma(1-\beta+\nu)}. \quad \text{[A1.3]}$$

To ensure at least one interior solution, we must impose the conditions under which $P(u^c)\leq 0$. For that purpose, we first compute that

$$P(u^c) = (1-\beta)(1-u^c)^{-\alpha}(\gamma-(1-u^c)^{\alpha}\rho). \quad \text{[A1.4]}$$

From [A1.4] we observe that $P(u^c)\leq 0$ if and only if $\gamma \leq (1-u^c)^{\alpha}\rho$. Combining this last condition with [A1.3] we obtain

$$\gamma\left(1+\frac{\nu-\sigma(1-\beta+\nu)}{\alpha(1-\beta)}\right)^{\alpha} \leq \rho. \quad \text{[A1.5]}$$

Hence, when [A1.5] holds with strict inequality, the equation $P(u)=0$ has two interior solutions u_1^* and u_2^*, with $u_1^* > u^c > u_2^*$. Whereas, if [A1.5] holds with equality, then $P(u)=0$ has a unique interior solution $u^*=u^c$.

Case B. Assume $P(0) < 0$. If $\nu - \sigma(1-\beta+\nu) \leq 0$, then $P(u)$ is an increasing function, so that there exists a unique u^* for which $P(u^*) = 0$. On the other hand, if $v - \sigma(1-\beta+\nu) > 0$, there also exists a unique u^* with $P(u^*) = 0$ since in this situation $P(u)$ is a quasiconvex function.

Case C. Assume $P(0) = 0$. If $\nu - \sigma(1-\beta+\nu) \leq 0$, then $P(u)$ is an increasing function, so that there does not exist any interior solution for $P(u) = 0$. On the other hand, if $\nu - \sigma(1-\beta+\nu) > 0$, there exists a unique u^* in $(0,1)$ for which $P(u^*) = 0$ since in this situation $P(u)$ is a quasiconvex function.

Appendix A2. Proof of Proposition 3

For the purpose of this appendix, we analyze the eigenvalue structure of the Jacobian matrix corresponding to the linearization of the system [12]-[14] around a steady-state equilibrium. The dimension of the unstable manifold is fully given by the number of eigenvalues with positive eigenvalues. Hence, we first characterize this Jacobian matrix, which takes the form

$$J = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix}, \qquad \text{[A2.1]}$$

whose elements are the partial derivatives of [12], [13] and [14] with respect to z, x and u, i.e.,

$$\begin{aligned}
a_{11} &= \left.\frac{\partial \dot{z}}{\partial z}\right|_{(z^*,x^*,u^*)} = (\beta-1)\,A\,(z^*)^{\beta-1}\,(u^*)^{1-\beta} < 0, \\
a_{12} &= \left.\frac{\partial \dot{z}}{\partial x}\right|_{(z^*,x^*,u^*)} = -z^* < 0, \\
a_{13} &= \left.\frac{\partial \dot{z}}{\partial u}\right|_{(z^*,x^*,u^*)} = -\frac{z^*}{u^*}\left[a_{11} + \frac{\gamma(1-\beta+\nu)}{\beta-1}(1-\alpha)u^*(1-u^*)^{-\alpha}\right] \\
a_{21} &= \left.\frac{\partial \dot{x}}{\partial z}\right|_{(z^*,x^*,u^*)} = \frac{\beta-\sigma}{\sigma}\frac{x^*}{z^*}a_{11}, \\
a_{22} &= \left.\frac{\partial \dot{x}}{\partial x}\right|_{(z^*,x^*,u^*)} = x^* > 0, \\
a_{23} &= \left.\frac{\partial \dot{x}}{\partial u}\right|_{(z^*,x^*,u^*)} = -\frac{\beta-\sigma}{\sigma}\frac{x^*}{u^*}a_{11}, \\
a_{31} &= \left.\frac{\partial \dot{u}}{\partial z}\right|_{(z^*,x^*,u^*)} = 0,
\end{aligned}$$

$$a_{32} = \left.\frac{\partial \dot{u}}{\partial x}\right|_{(z^*,x^*,u^*)} = -\frac{\beta u^*(1-u^*)}{\alpha u^* + \beta(1-u^*)} < 0,$$

$$a_{33} = \left.\frac{\partial \dot{u}}{\partial u}\right|_{(z^*,x^*,u^*)} = \frac{\gamma(1-\alpha)u^*(1-u^*)^{-\alpha}}{\alpha u^* + \beta(1-u^*)}[\alpha u^* + (\beta-\nu)(1-u^*)],$$

where (z^*, x^*, u^*) is one of the interior steady-state equilibria found in Section 4.

We now compute the sign of the eigenvalues of the Jacobian matrix [A2.1] in order to characterize the local stability of the system. These eigenvalues are the roots μ_i of the following characteristic polynomial:

$$-\mu^3 + Tr(J)\mu^2 - B(J)\mu + Det(J) = 0, \qquad \text{[A2.2]}$$

where $Tr(J)$ and $Det(J)$ respectively denote the trace and the determinant of the Jacobian matrix [A2.1], and

$$B(J) = a_{12}a_{22} + a_{13}a_{31} + a_{23}a_{32} - (a_{11}a_{22} + a_{11}a_{33} + a_{22}a_{33}). \qquad \text{[A2.3]}$$

Furthermore, the following four lemmas characterize these three coefficients of the characteristic polynomial [A2.2].

LEMMA 2: *Det* (J) is positive when $P'(u^*) < 0$, whereas $Det(J)$ is negative if $P'(u^*) > 0$.

PROOF: Using the matrix elements a_{ij} computed above, and after a simple manipulation, we can rewrite $Det(J)$ as follows:

$$Det(J) = a_{11}x^* \frac{\gamma(1-\alpha)u^*(1-u^*)^{1-\alpha}}{(\beta-1)(\alpha u^* + \beta(1-u^*))} \left((\nu - \sigma(1-\beta+\nu)) - \frac{(1-\beta)\alpha u^*}{(1-u^*)}\right).$$

Since a_{11} is negative, we observe from [A1.2] that $Det(J)$ has the opposite sign of $P'(u^*)$. Thus, the lemma is established. Q.E.D.

LEMMA 3: *$Tr(J)$ is negative if and only if $2\alpha u^* + (2\beta - \nu)(1-u^*) < 0$.*

PROOF: Replacing a_{11}, a_{22} and a_{33} by their values, $Tr(J)$ transforms into

$$Tr(J) = \frac{\gamma(1-\alpha)u^*(1-u^*)^{-\alpha}}{\alpha u^* + \beta(1-u^*)}\left[2\alpha u^* + (2\beta-\nu)(1-u^*)\right].$$

Thus, the lemma follows directly. Q.E.D.

LEMMA 4: *$B(J)$ is negative if $\alpha u^* + (\beta - \nu)(1-u^*) < 0$.*

PROOF: Substituting in [A2.3] for the elements a_{ij} computed above, and after a simple manipulation, we can rewrite

$$B(J) = \frac{\beta(\alpha u^* + \sigma(1-u^*))}{\sigma(\alpha u^* + \beta(1-u^*))} a_{11} x^* + (a_{11} + x^*) a_{33}.$$

Moreover, from [15] to [17] we can check that $a_{11} + x^* = \gamma(1-\alpha)u^*(1-u^*) > 0$. Hence, since a_{11} is negative, $a_{33} < 0$ is a sufficient condition for $B(J)$ to be negative. Therefore, the result is proved by taking the expression of a_{33} given above in this appendix. Q.E.D.

LEMMA 5: *A sufficient condition for $B(J)$ to be negative is that $Tr(J) < 0$.*

PROOF: It follows directly from Lemmas 3 and 4. Q.E.D.

The characteristic polynomial [A2.2] has three roots, two of them probably complex but given in a conjugate form. To determine the sign of the real part of these roots, we will use Routh's theorem.[13] In our particular case, this theorem says that the number of roots of [A2.2] with positive real parts is equal to the number of variations of sign in the sequence: $-1, Tr(J), -B(J) + (Det(J)/Tr(J)), Det(J)$. From now on this sequence will be called the Routh Sequence.[14] Hence, we can now prove Proposition 3.

PROOF OF PART (i). From Appendix A.1 we deduce that the derivative of $P(u)$ is positive at (z^*, x^*, u^*). Hence, the determinant of the Jacobian matrix evaluated at (z^*, x^*, u^*) is negative from Lemma 2. The number of eigenvalues of [A2.1] with positive real parts is then given by the sign of $Tr(J)$. If $Tr(J)$ is positive, there are always two variations of sign in the Routh Sequence regardless of the sign of $-B(J) + (Det(J)/Tr(J))$. If $Tr(J)$ is negative, $-B(J) + (Det(J)/Tr(J))$ is positive by Lemma 5, and then there are also two variations of sign in the aforementioned sequence. Therefore, the Jacobian matrix [A2.1] evaluated at (z^*, x^*, u^*) always has one negative eigenvalue and two eigenvalues with positive real parts.

PROOF OF PART (ii). From Appendix A1, we observe that $P'(u_1^*) > 0$ and $P'(u_2^*) < 0$. Hence, we can first conclude, following the proof of part (i), that the Jacobian matrix evaluated at (z_1^*, x_1^*, u_1^*) always has one negative eigenvalue and two eigenvalues with positive real parts. Second, the determinant of the Jacobian matrix [A2.1] evaluated at (z_2^*, x_2^*, u_2^*) is positive as Lemma 2 establishes. Thus, the number of eigenvalues of [A2.1] evaluated

[13] See Gantmacher (1960, chapter XV).
[14] This sequence is actually the first column of Routh's Scheme corresponding to the polynomial [A2.2].

at (z_2^*, x_2^*, u_2^*) with positive real parts is given by the sign of $Tr(J)$. If $Tr(J)$ is negative, one variation of sign in the Routh Sequence occurs regardless of the sign of $-B(J) + (Det(J)/Tr(J))$. On the other hand, if $Tr(J)$ is positive, two cases are possible. (i) When $B(J)$ is negative, $-B(J) + (Det(J)/Tr(J))$ is positive. Thus, in this case, the Jacobian matrix [A2.1] evaluated at (z_2^*, x_2^*, u_2^*) has one positive eigenvalue and two eigenvalues with negative real parts. (ii) When $B(J)$ is positive, $-B(J) + (Det(J)/Tr(J))$ can be either positive or negative. Therefore, the Jacobian matrix [A2.1] evaluated at (z_2^*, x_2^*, u_2^*) can have either one positive eigenvalue and two eigenvalues with negative real parts or three eigenvalues with positive real part.

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Abstract

Este trabajo muestra que múltiples tasas de crecimientos a largo plazo, las cuales son globalmente indeterminadas, pueden surgir fácilmente en el modelo de crecimiento con dos sectores introducido por Lucas (1988). Este resultado está generado por la existencia de rendimientos privados decrecientes en tiempo en el proceso de acumulación de capital humano en la producción. El trabajo sostiene que surgen dos sendas interiores de crecimiento equilibrado bajo una elasticidad de sustitución intemporal lo suficientemente grande. Una de esas sendas es localmente determinada, mientras que la otra puede ser localmente indeterminada. Además, mostramos que esas sendas de crecimiento equilibrado pueden ser también globalmente indeterminadas.

Palabras clave: Crecimiento endógeno, capital humano, externalidades, equilibrio múltiple, indeterminación.

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