

INCIDENCE OF NO SHOW PARADOXES IN CONDORCET CHOICE FUNCTIONS

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The incidence of some paradoxes related to the so-called No Show Paradox is analyzed in the context of Condorcet choice functions. To do this, two Monotonicity properties in the face of new voters and a version of Moulin's Participation property for the Social Choice framework are defined. It is proved that the paradoxes related to Monotonicity affect large families of choice functions, while the paradox related to Participation affects all the Condorcet choice functions.

1. Introduction

The search for an optimal choice or voting method was shaken by the publication, in 1951, of the Arrow Impossibility Theorem, which led to a fast development of the subject. This theorem, and its subsequent developments and refinements, could be interpreted, in a context of choice (of candidates or of alternative options), as saying that no choice method or rule simultaneously fulfils some minimal properties that seem to be required by any reasonable method. In other words, no choice method is free from paradoxes (failures to satisfy some intuitively compelling properties).

As a consequence of this result, most of the research on the subject has followed the axiomatic approach. An important line of investigation tried to reach a Possibility result by weakening the axioms of the Arrow Theorem. See Villar (1988) for an instructive review.

This paper belongs to a different, but related, line of research, which has developed a systematic study of methods and method families, as well as analyzed the incidence of paradoxes in some families. Well-known examples in this line are the axiomatic characterization of the positional functions in Young (1975), the Borda function in Young (1974), and the Kemeny function in Young and Levenglick (1978). Other examples are the detailed

* I wish to thank Carol Warren for his kind help with the English. I wish also to thank two anonymous referees for their valuable comments and suggestions, in particular for Definition 7.

analysis of a wide group of Condorcet choice functions in Fishburn (1977), and the result in Moulin (1988a) establishing that all Condorcet voting rules are affected by the No-Show paradox.

In this paper, which is concerned with choice functions, we propose three paradoxes in the same spirit of the No Show paradox, and three intuitively appealing properties (two weakening of the Duality property defined in Fishburn (1973), and one Conservation property). The incidence of these three paradoxes in the family of Condorcet choice functions is analyzed, and four propositions that establish that fulfilling some of the above properties implies a paradox are proved. We also establish that there is a function satisfying some of the strongest defined properties which, for three or less candidates, is free from these paradoxes.

The interest and relevance of Condorcet functions in choice or voting relies on its fidelity to the democratic principle which asserts that if a candidate exists that is preferred to any other by a majority of voters, this candidate should be the only one chosen. However, as will be shown below, *all* Condorcet functions necessarily have some undesirable and paradoxical features, one of which is the above mentioned No Show paradox, which means that a voter might do better by abstaining, that is to say, the candidate chosen if the voter abstains is more acceptable to him than the candidate chosen if the voter had submitted a ballot with his true preferences.

The occurrence of this paradox means that a desirable property, which Moulin (1988a) calls Participation, fails to be satisfied. This property has some relation with Young's Consistency, which is essential in positional functions. The Consistency property, which is also incompatible with all Condorcet functions, means that if some candidate is chosen by an electoral body E_1 , and also by a disjoint electoral body E_2 , those and only those candidates chosen by the two bodies will be chosen by the joint body $E_1 \cup E_2$.

Section 2 presents the basic terminology, defines the properties related with the already known No Show paradoxes, and briefly reviews the main known results. Section 3 defines the new paradoxes and the Duality and Conservation properties, and proves five propositions that establish which properties imply the occurrence of paradoxes. Section 4 summarizes and makes some additional remarks.

2. Basic terminology and known results

The terminology of Fishburn (1977) will be used for the formal setting and for the definition of Condorcet properties. When a particular function is alluded to, the definition of Fishburn (1977) will also be used.

Let $X = \{x_1, x_2, \dots, x_n\}$ be a finite set with two or more candidates. Preferences of any voter take the form of a preorder over X . For simplicity, we suppose that there are no ties in the preorder, which then becomes a (strict and complete) order.

Given the set of candidates $X = \{x_1, x_2, \dots, x_n\}$, and any finite set $V = \{1, 2, \dots, m\}$ with one or more voters, we call a *Situation* any pair (X, p) , where p is a preference profile over X from V , that is to say, a m -tuple of orders over X , each one meaning the preferences of a voter over X .

We call *Social Choice Function* (SCF from now on) any function f which maps any situation (X, p) to a non-empty subset of X , $f(X, p)$. The elements of $f(X, p)$ are candidates chosen over X from the preference profile p . If $f(X, p)$ is assumed to have only one element, we say that f is a *Voting Rule*.

Given any X , any two disjoint sets of voters $V_1 = \{1, 2, \dots, m_1\}$ and $V_2 = \{m_1 + 1, m_1 + 2, \dots, m_1 + m_2\}$, and any two preference profiles p_1 and p_2 over X from, respectively, V_1 and V_2 , we can combine these two profiles in order to obtain a new profile over X , but now from $V_1 \cup V_2$. This new profile will be called $p_1 + p_2$.

Given any situation (X, p) and any two different candidates x, y from X , $p(x, y)$ means the number of voters in profile p which prefer x to y . Because there are no ties, $p(x, y) + p(y, x)$ is the total number of voters in profile p . The square matrix M_p , whose order is the number of candidates and whose entries are $p(x, y)$, will be called *Paired Preferences Matrix* for (X, p) . In this matrix, for any given candidate there is a row (say the i -th row) and a column (the i -th column). The entries of the main diagonal, $p(x, x)$, will be left empty, as usual.

For a given situation (X, p) , the *Dual situation* $(X, p)^*$ results from inverting the preferences of every voter in p .

For a given situation (X, p) and a given candidate $x \in X$, the *Borda count* of x is $BC((X, p), x) = \sum_{y \in X} p(x, y)$, obtained summing up all the entries of the row corresponding to x in M_p . A candidate x is called a *Borda's average* if its Borda count is the average (arithmetic mean) of all the Borda counts in (X, p) . This is equivalent to saying that the result of summing up the entries of the x row ($\sum_{y \in X} p(x, y)$) is the same as summing up the entries of the x column ($\sum_{y \in X} p(y, x)$). A candidate x is called a *Generalised Borda's average* if the difference between the sum of the x row and the sum of the x column, in absolute terms, is minimal. Obviously, a Borda's average candidate is also a Generalised Borda's average. Although there are situations with no Borda's average candidate, a Generalised Borda's average candidate (perhaps many) always exists.

Given (X, p) , with m voters, we say that x *defeats* y *by a majority* for profile p , if $p(x, y) > p(y, x)$, and we say that x *ties* y *by a majority* for profile p , if $p(x, y) = p(y, x)$. A candidate x which defeats (defeats or ties) any other by a majority, is called a *Condorcet (QuasiCondorcet)* candidate.

2.1. Condorcet Properties

Definition 1: A SCF f is called *Condorcet* if and only if for every situation (X, p) , the *Condorcet Principle* holds: If $x \in X$ and x defeats any $y \in X - \{x\}$ by a majority, then x is the only chosen candidate.

Definition 2: A SCF f is called *Inclusive Condorcet* if and only if it is Condorcet, and for any situation (X, p) , if QuasiCondorcet candidates exist, all of them are chosen.

Definition 3: A SCF f is called *Exclusive Condorcet* if and only if for any given situation (X, p) , any chosen candidate is QuasiCondorcet, if QuasiCondorcet candidates exist.

A SCF is called *Strict Condorcet* when it is Inclusive Condorcet and Exclusive Condorcet.

It is obvious that Strict Condorcet is the strongest of these four properties, and that Condorcet is the weakest one. Exclusive Condorcet and Inclusive Condorcet are independent. When there are QuasiCondorcet candidates, the Inclusive Condorcet methods will chose all the QuasiCondorcet candidates, perhaps with other candidates (if a Condorcet candidate exists, they will chose only it), Exclusive Condorcet methods will only chose QuasiCondorcet candidates, perhaps not all of them, and Strict Condorcet methods will chose all QuasiCondorcet candidates, and only them.

2.2. Some known results with No Show Paradoxes

Definition 4: A SCF f satisfies the *Consistency property* if and only if for any given pair of situations (X, p_1) and (X, p_2) , $f(X, p_1) \cap f(X, p_2) \neq \emptyset$ implies $f(X, p_1 + p_2) = f(X, p_1) \cap f(X, p_2)$.

In words, if some candidates are chosen for profile p_1 and profile p_2 , they, and only they, are chosen when the two profiles are combined. This property characterizes, along with Anonymity and Neutrality, the positional choice functions, whose best known examples are the *Plurality rule* (the chosen candidates are those who are the most preferred by a highest number of voters) and the *Borda rule* (the chosen candidates are those who have the highest *Borda count*). See Young (1975), and see also Young (1974) for a characterization of the Borda rule where the Consistency property plays a fundamental role.

The Consistency property could be interpreted as a kind of Monotonicity property in the face of combinations of profiles because, when candidate x is chosen for profile p_1 and the same one is also observed to be chosen for profile p_2 , it seems reasonable to expect that adding this new profile will not hurt x , and so x will continue to be chosen and no other candidate not chosen before will be chosen now.

The following property, from Moulin (1988a), is defined in the context of Voting rules (for any situation, only one candidate is chosen).

Definition 5: A Voting rule f satisfies the *Participation property* if and only if for any given pair of situations (X, p) and (X, v) , in which profile v has only one voter, if $f(X, p) = \{x\}$ and x is preferred to y in v , $f(X, p + v) \neq \{y\}$.

That is to say, if x is the only candidate chosen for a situation and a new voter is added who prefers x to y , it cannot be the case that y will be the only candidate chosen in the new situation. From the point of view of the new voter, it would do better if he abstained, because submitting his ballot would result in the election of a less preferred candidate. If we apply Moulin's terminology, failing to satisfy Participation invokes the *No Show paradox*.

The logical relations between these two properties and the Condorcet properties are shown in Propositions 1 and 2 below, established respectively in Young and Levenglick (1978) and Moulin (1988a). See also Moulin (1988b, chap. 9).

Proposition 1: No Condorcet SCF satisfies the Consistency property.

Proposition 2: No Condorcet Voting Rule satisfies the Participation property.

Because they belong to different formal frameworks, Consistency and Participation properties are not logically related. Moreover, Moulin (1988a) proved that Participation does not imply nor is implied by the Reinforcement property, which is a natural translation of Consistency to the Voting Rules framework. Although Moulin says Participation is an intuitively weaker property than Reinforcement, something that would make the occurrence of the No Show Paradox intuitively worse than failing to satisfy Consistency, we think that, at least from the point of view of Monotonicity, this opinion is questionable because the candidate x chosen for (X, p) does not necessarily improve his global performance when the new voter is added.

3. New paradoxes

The following definition tries to delimit, in the SCF context, two properties which, having a clear monotonic meaning, are logically weaker than Consistency, and intuitively as weak as Participation. The failure by an SCF f to satisfy any of these two new properties means that f falls in the corresponding No Show paradox.

Definition 6: A SCF f satisfies the *Monotonicity property in the face of new voters* if and only if for any given pair of situations (X, p) and (X, v) , where profile v has only one voter, if $x \in f(X, p)$ and x is preferred to any y in v , then $x \in f(X, p + v) \subset f(X, p)$. If the inclusion of $f(X, p + v)$ in $f(X, p)$ is not required, we will call this the *Weak Monotonicity property in the face of new voters*.

In words, if candidate x is chosen for a situation and a new voter is added who prefers x to any other:

- a) x will remain chosen, in Weak Monotonicity.
- b) x will remain chosen and no one not chosen before will be chosen now, in Monotonicity.

It is clear for Condorcet SCFs that the Consistency property would imply, with the hypotheses made in Definition 6, that x was the only candidate chosen when the new voter was added. Hence, *Consistency implies Monotonicity in the face of new voters*, for Condorcet SCFs. Furthermore, the monotonic content of these two properties seems clear: Any chosen candidate x improves in the global set of preferences if a new voter is added for whom x is the favourite, so that x should not be hurt, by not being chosen, nor by having to share the choice set with a candidate not chosen before.

We must say that Condorcet SCFs which satisfy these two monotonicity properties do exist. Consider, for example, the *Maximin* SCF (by which, those candidates are chosen whose minimal entry in their corresponding M_p row is maximal). When a new voter v is added (with preferences $xyz\dots$) to the situation (X, p) the new Paired Preferences Matrix M_{p+v} is obtained from M_p adding one unit to the following entries: every entry $p(x, t)$ of the x row; every entry $p(y, t)$ of the y row, except $p(y, x)$; every entry $p(z, t)$ of the z row, except $p(z, x)$ and $p(z, y)$; and so on. It is clear that if the minimal entry of the x row is maximal in M_p , the minimal entry of the x row will also be maximal in M_{p+v} . Furthermore, if the minimal entry of any other row is not maximal in M_p , so it will not be maximal in M_{p+v} . Thus, the Maximin SCF satisfies both properties. However, as will be shown below, the failure to satisfy these properties is a frequent occurrence with Condorcet SCFs.

The following definition tries to translate a weak form of the Participation property to the Social Choice framework.

Definition 7: A SCF f satisfies the *Choice Participation* property if and only if for any given pair of situations (X, p) and (X, v) , where profile v has only one voter, if $x \in f(X, p)$ and x is preferred to y in v , then $(y \in f(X, p+v))$ implies $x \in f(X, p+v)$.

In words, if candidate x is chosen for a situation and a new voter is added who prefers x to y , candidate y will not be chosen if it is not accompanied by candidate x . This property is obviously stronger than Weak Monotonicity in the face of new voters.

3.1. Three impossibility results for some families of Condorcet SCFs

The following properties allow to identify families of Condorcet choice functions which fail to satisfy some of the two monotonicity properties defined above.

Definition 8: A SCF f satisfies the *Moderate Duality* property if and only if for any given situation (X, p) , $\{x\} = f(X, p) \cap f(X, p)^*$ implies x is a Generalised Borda's average.

In words, only a Generalised Borda's average candidate can be the unique candidate simultaneously chosen for a situation and for its dual.

Definition 9: A SCF f satisfies the *Weak Duality* property if and only if for any given situation (X, p) , $f(X, p) = \{x\}$ and $f(X, p)^* = \{x\}$ implies x is a Generalised Borda's average.

In words, only a Generalised Borda's average candidate can be the unique candidate chosen for a situation and the unique candidate chosen for its dual.

These two Duality properties are intended as reasonable weakening of the Duality property defined in Fishburn (1973), which excluded any candidate from being chosen for a situation and for its dual, unless all candidates were chosen. On p. 158 Fishburn admitted that the proposed definition of Duality was unnecessarily restrictive in some cases. The normative idea embodied by Duality is that when a candidate gets a good global performance (that allows him to be chosen) for a set of preference orders, his performance should become poor (excluding him from being chosen) when these orders are inverted.

However, such an idea loses its normative strength when we consider situations in which the relative performances of candidates are not clear. In these situations it may be reasonable to choose the more balanced candidates; those whose performance will only be slightly altered when preference orders are inverted. Nevertheless, besides its relative normative value, Duality properties have a significant descriptive value.

Definition 10: Given a situation (X, p) and two candidates x and y , we say x *Dominates y through Compensation* if and only if the three following conditions hold:

- a) $p(x, y) > p(y, x)$
- b) $p(x, z) > p(y, z)$ for any $z \in X - \{x, y\}$ except perhaps for a unique z
- c) If $p(x, z) < p(y, z)$, then $p(x, y) - p(y, x) > p(y, z) - p(x, z)$

In words, x defeats y by a majority, x performs better than y in his confrontation with any other candidate (except perhaps with the z candidate), and the difference in favour of x in his confrontation with y more than compensates for the difference in favour of y when both are confronted with z .

Definition 11: A SCF f *Conserves Domination through Compensation* if and only if for any given situation (X, p) , x Dominates y through Compensation implies $f(X, p) \neq \{y\}$.

So, the condition imposed on f is to not choose only a candidate who is dominated through compensation.

The three following lemmas will be needed in the proof of Propositions 3, 4 and 5 below. We denote $xyz\ldots$ the preference order in which x is the most preferred candidate, y is the second one, and so on.

Lemma 1: Given any Inclusive Condorcet SCF f which is Monotonic in the face of new voters, any situation (X, p) and any two candidates x and z , if $\text{Min}_{y \in X} p(x, y) = p(x, z) = \text{Min}_{y \in X} p(z, y)$, then $(x \in f(X, p))$ implies $z \in f(X, p)$.

Lemma 2: Given any Exclusive Condorcet SCF f which is Weak Monotonic in the face of new voters, any situation (X, p) and any two candidates x and z , if $\text{Min}_{y \in X} p(x, y) < p(x, z) = \text{Min}_{y \in X} p(z, y)$, then $x \notin f(X, p)$.

Lemma 3: Given any Condorcet SCF f which is Weak Monotonic in the face of new voters, any situation (X, p) and any two candidates x and z , if $p(x, z) < \text{Min}_{y \in X} p(z, y)$, then $x \notin f(X, p)$.

Proof of Lemma 1: Assume $\text{Min}_{y \in X} p(x, y) = p(x, z) = \text{Min}_{y \in X} p(z, y)$, and suppose x is chosen for (X, p) . Observe that when a new voter v is added (with preferences $abc\dots$) to the situation (X, p) , the new Paired Preferences Matrix M_{p+v} is obtained from M_p adding one unit to the following entries: every entry $p(a, t)$ of the a row: every entry $p(b, t)$ of the b row, except $p(b, a)$; every entry $p(c, t)$ of the c row, except $p(c, a)$ and $p(c, b)$; and so on.

Let us add to profile p , iteratively, a number $h = p(z, x) - p(x, z)$ of new voters, all of them with identical preference order $xz\dots$. Call p' the new profile. The new number of voters is $m + h$, where m is the number of voters in profile p . The new minimal entry in the x row is $p'(x, z) = p(x, z) + h = p(z, x)$. The new minimal entry in the z row is $p'(z, x) = p(z, x)$, because $\text{Min}_{y \in X} p(z, y) + h = p(x, z) + h = p(z, x) = p'(z, x)$. Since $p'(z, x) = (m + h) / 2$ (because $p'(z, x) = p'(z, x)$), both x and z are QuasiCondorcet candidates in the new situation. As f is supposed to be Inclusive Condorcet, both candidates will be chosen for the new situation (X, p') .

This implies that z is chosen for (X, p) , because in any other case, as the new h voters are added, it would necessarily occur at some point that z will become chosen, and this would contradict the Monotonicity property in the face of new voters. \square

Proof of Lemma 2: Assume $\text{Min}_{y \in X} p(x, y) < p(x, z) = \text{Min}_{y \in X} p(z, y)$. Call m the number of voters in profile p . As above, let us add to profile p , iteratively, a number $h = p(z, x) - p(x, z)$ of new voters, all sharing an identical preference order $xz\dots$. Call p' the new profile. The new minimal entry in the x row is strictly lower than $p'(x, z) = p(x, z) + h = p(z, x)$, and the new minimal entry of the z row is $p'(z, x) = p(z, x)$, because $\text{Min}_{y \in X} p(z, y) + h = p(x, z) + h = p(z, x) = p'(z, x)$. Moreover, $p'(z, x) = (m + h) / 2$ (because $p'(z, x) = p'(x, z)$), so that z is a QuasiCondorcet candidate in the new situation, but x is not. Since f is supposed to be Exclusive Condorcet, x is not chosen for the new situation (X, p') .

This implies that x is not chosen for (X, p) , because in any other case, as the new h voters are added, it would necessarily occur at some point that x will become not chosen, so contradicting the Weak Monotonicity property in the face of new voters. \square

Proof of Lemma 3: What follows is an adaptation of the proof of an analogous result in Moulin (1988a, p. 57). Let us suppose $p(x, z) < \text{Min}_{y \in X} p(z, y)$. Call m the number of voters in profile p . Iteratively adding to p a number $h = p(z, x) - \text{Min}_{y \in X} p(z, y)$ of new voters, all with identical preference order $xz\dots$, the minimal entry of the z row in the new profile p' is $p'(z, x) = p(z, x)$. On the other hand, $p'(x, z) < p'(z, x)$, because $p'(x, z) = p(x, z) + h = p(x, z) + p(z, x) - \text{Min}_{y \in X} p(z, y) = p(z, x) + [p(x, z) - \text{Min}_{y \in X} p(z, y)] < p(z, x) = p'(z, x)$. Hence, the minimal entry on the z row in profile p' ; $p'(z, x)$, is higher than $(m + h)/2$, making z a Condorcet candidate in the new situation and, because of the supposed Condorcet property, the only candidate chosen. Reasoning as in Lemma 2, the proof is completed. \square

Proposition 3: No Inclusive Condorcet and Moderate Dual SCF is Monotonic in the face of new voters.

Proof: Let $X = \{x, y, z, t\}$, p be the following profile and M_p its corresponding Paired Preferences Matrix:

8 voters: $zyxt$		x	y	z	t
5 voters: $tzyx$		x	7	6	12
p : 4 voters: $txyz$	M_p :	y	15	8	10
2 voter: $yzxt$		z	16	14	10
2 voter: $xtyz$		t	10	12	12
1 voters: $tzxy$					

Let us suppose f is Monotonic in the face of new voters and Inclusive Condorcet. Applying Lemma 3 to the pairs (x, y) and (y, z) , candidates x and y are not chosen for (X, p) . From Lemma 1 applied to the pair (z, t) , if z is chosen for (X, p) , t is also chosen. So, t is necessarily chosen for (X, p) , perhaps with z .

The dual situation $(X, p)^*$ is the following:

8 voters: $txyz$		x	y	z	t
5 voters: $xyzt$		x	15	16	10
p^* : 4 voters: $zyxt$	M_{p^*} :	y	7	14	12
2 voter: $txzy$		z	6	8	12
2 voter: $zytx$		t	12	10	10
1 voters: $yxzt$					

Applying Lemma 3 to the pairs (y, x) and (z, x) , candidates y and z are not chosen for $(X, p)^*$. From Lemma 1 applied to (x, t) , if x is chosen, t is also

chosen. Therefore, t is necessarily chosen for $(X, p)^*$, perhaps with x . Consequently, t is the unique candidate who is chosen simultaneously for (X, p) and $(X, p)^*$. Since candidate t is not a Generalised Borda's average, f fails to satisfy the Moderate Duality property, thus completing the proof. \square

Proposition 4: No Strict Condorcet and Weak Dual SCF is Monotonic in the face of new voters.

Proof: Let $X = \{x, y, z, u, t\}$, p be the following profile, and M_p its corresponding Paired Preferences Matrix:

4 voters: $uzyxt$	x	y	z	u	t
2 voters: $tzxyu$	x	4	5	8	7
p : 2 voters: $txuzy$	M_p : y	8	4	6	7
1 voter: $tyxuz$	z	7	8	4	6
1 voter: $yxtuz$	u	4	6	8	5
1 voter: $yxztu$	t	5	5	6	7
1 voter: $yzxut$					

Suppose f is Monotonic in the face of new voters and Strict Condorcet.

From Lemma 2 applied to (u, t) , candidate u is not chosen for (X, p) . From Lemma 1 applied to (z, u) , if candidate z is chosen for (X, p) , u will also be chosen. As a consequence, z is not chosen. From Lemma 1 applied to (y, z) , if candidate y is chosen, z is also chosen. As a consequence, y is not chosen. From Lemma 1 applied to (x, y) , if candidate x is chosen, y is also chosen. As a consequence, x is not chosen.

Thus, t is the unique candidate chosen for (X, p) . It is clear that for the dual situation $(X, p)^*$, the last reasoning can be repeated step by step (candidates t, x, y, z, u now respectively play the role taken before by t, u, z, y, x), so proving that candidate t is the only one chosen for $(X, p)^*$. Because t is not a Generalised Borda's average, f fails to satisfy the Weak Duality property, thus completing the proof. \square

Proposition 5: No Condorcet SCF which Conserves Domination through Compensation, is Weak Monotonic in the face of new voters.

Proof: Let $X = \{x, y, z, t\}$, p the following profile, and M_p its corresponding Paired Preferences Matrix:

9 voters: $yxtz$	x	y	z	t
8 voters: $zytx$	x	10	21	19
p : 5 voters: $xzty$	M_p : y	23	11	19

5 voters: $x t z y$	z	12	22	13
4 voters: $t z y x$	t	14	14	20
2 voters: $y t x z$				

Suppose f is Condorcet and Weak Monotonic in the face of new voters. From Lemma 3 applied to the pairs (x, y) , (y, z) and (z, t) , candidates x, y and z are not chosen for (X, \hat{p}) , and thus candidate t is the only one chosen. However, x Dominates t through Compensation, because $\hat{p}(x, z) = 21 > \hat{p}(t, z) = 20$ and also $\hat{p}(x, t) - \hat{p}(t, x) > \hat{p}(t, y) - \hat{p}(x, y)$. Therefore, f fails to satisfy the Conservation of Domination through Compensation property, thus completing the proof. \square

3.2. An impossibility result for all Condorcet SCFs

Proposition 6: No Condorcet SCF satisfies the Choice Participation property.

The following proof is an adaptation of the proof of Proposition 2, made in Moulin (1988a, pp. 57-59).

Proof: Let $X = \{x, y, z, t\}$, \hat{p} the following profile, and $M_{\hat{p}}$ its corresponding Paired Preferences Matrix:

5 voters: $t y z x$	x	y	z	t
4 voters: $y z x t$	x	6	6	10
\hat{p} : 3 voters: $x t z y$	$M_{\hat{p}}$: y	9	12	4
3 voters: $x t y z$	z	9	3	4
	t	5	11	11

Suppose f is Condorcet and satisfies Choice Participation. Since Choice Participation implies Weak Monotonicity in the face of new voters, Lemma 3 can be applied to the pairs (z, y) , (y, t) and (t, x) to conclude that candidates z, y and t are not chosen for (X, \hat{p}) . Thus, candidate x is chosen alone.

If we add iteratively 4 voters with preferences $z x y t$, the final Paired Preferences Matrix is:

	x	y	z	t
	x	10	6	14
$M_{\hat{p} + v + v + v + v}$:	y	9	12	8
	z	13	7	8
	t	5	11	11

Applying again Lemma 3 to the pairs (t, x) , (x, z) and (z, y) it is immediate to see that candidates t , x and z are not chosen for $(X, p + v + v + v + v)$, implying that y is the unique candidate chosen now. Therefore, while the new four voters are being added, candidate x is not chosen at some point, contradicting the Weak Monotonicity property in the face of new voters, and consequently the Choice Participation property. \square

3.3. A possibility result for three candidates

The following proposition shows that the logical incompatibilities established in Propositions 3, 4 and 5 disappear for the case in which there are at most three candidates.

Proposition 7: There exists a Strict Condorcet and Moderate Dual social choice function that Conserves Domination through Compensation such that, for the case in which X has three or less candidates, it satisfies the Monotonicity property in the face of new voters and the Choice Participation property.

Proof: Let f be the following SCF: «Given any situation (X, p) , if some QuasiCondorcet candidates exist, they and only they are chosen, and if no QuasiCondorcet candidates exist, the Kemeny function is applied».

First, let us recall the definition of the Kemeny function, and prove that it is Moderate Dual (Lemma 4) and Conserves Domination through Compensation (Lemma 5):

Given a preference order r on X , define

$$r(x, y) = \begin{cases} 1 & \text{if } x \text{ is preferred to } y \text{ in order } r \\ 0 & \text{otherwise.} \end{cases}$$

Given any situation (X, p) , candidate x is chosen according to the Kemeny function if and only if there exists an order r on X whose most preferred candidate is x , and in which the sum $\sum_{x, y \in X} p(x, y) r(x, y)$ (called the Kemeny score of r , and also denoted $K[(X, p), r]$) is maximal over the set of orders on X .

In words that refer to the row and column arrangement of M_p , x is chosen if and only if an order r on X exists whose most preferred candidate is x , and in which, when reordering the rows and columns of M_p according to r , the sum of the entries above the main diagonal in M_p (the Kemeny score of r) takes a maximal value.

Fishburn (1977) has proven that this function is Inclusive Condorcet.

Lemma 4: The Kemeny function is Moderate Dual.

Proof of Lemma 4: Let (X, p) be a situation with $n + 1$ candidates, and $x \in f(X, p) \cap f(X, p)^*$. Since candidate x belongs to $f(X, p)$, there exists an order

$r: xa_1 \dots a_n$ which reaches a maximal Kemeny score for (X, ρ) , denoted $K[(X, \rho), r]$. Let us call this maximal score Max_x . Obvious reasons of symmetry imply that $r^*: a_n \dots a_1 x$ has a Kemeny score $K[(X, \rho)^*, r^*]$ for $(X, \rho)^*$ which is maximal and which equals Max_x . On the other hand, because candidate x also belongs to $f(X, \rho)^*$, there exists an order $s^*: xb_n \dots b_1$ such that $K[(X, \rho)^*, s^*]$ is maximal for $(X, \rho)^*$, thus equalling Max_x . Moreover, by a symmetric argument, $K[(X, \rho), s]$, where s is the order $b_1 \dots b_n x$, is maximal for (X, ρ) and also equals Max_x .

Using the fact that $K[(X, \rho), z_0 z_1 \dots z_n] = BC((X, \rho), z_0) + K[(X - \{z_0\}, \rho), z_1 \dots z_n]$, it is easy to deduce that the Kemeny scores $K[(X - \{x\}, \rho), r^*]$ and $K[(X - \{x\}, \rho)^*, s^*]$, corresponding to the orders $r^*: a_1 \dots a_n$ in $(X - \{x\}, \rho)$ and $s^*: b_n \dots b_1$ in $(X - \{x\}, \rho)^*$ are maximal for their respective situations. This implies that such scores are equal. Let us now compare the Kemeny scores of r and s^* , for (X, ρ) and for $(X, \rho)^*$, respectively.

$$\text{Max}_x = K[(X, \rho), r] = BC((X, \rho), x) + K[(X - \{x\}, \rho), r^*]$$

$$\text{Max}_x = K[(X, \rho)^*, s^*] = BC((X, \rho)^*, x) + K[(X - \{x\}, \rho)^*, s^{**}]$$

Therefore, $BC((X, \rho), x) = BC((X, \rho)^*, x)$, thus the sum of the x row in M_ρ is equal to the sum of the x column in M_ρ . Consequently, candidate x is a *Borda's average* for the situation (X, ρ) , thus completing the proof¹.

Lemma 5: The Kemeny function Conserves Domination through Compensation.

Proof of Lemma 5: Given any situation (X, ρ) , let us suppose that x Dominates y through Compensation and that y is the unique chosen candidate. This implies that there exists an order r on X whose most preferred candidate is y , and such that $K[(X, \rho), r]$ is maximal.

Let now s be the order which results from r by interchanging the positions of x and y . It is immediate to check that

$$K[(X, \rho), s] = K[(X, \rho), r] + \rho(x, y) - \rho(y, x) + \sum_{t \in T} \rho(x, t) - \sum_{t \in T} \rho(y, t),$$

where T is the set of candidates between x and y in the order r .

On the other hand, $\rho(x, y) - \rho(y, x) > \sum_{t \in T} \rho(y, t) - \sum_{t \in T} \rho(x, t)$, because only one summand $\rho(y, z)$ can be higher than the corresponding summand $\rho(x, z)$ and, by hypothesis, $\rho(x, y) - \rho(y, x) > \rho(y, z) - \rho(x, z)$. So, $K[(X, \rho), s] > K[(X, \rho), r]$, thus contradicting the supposed fact of y being chosen, and so completing the proof.

Let us now proceed to the rest of the proof of proposition 7. It is obvious that f is Strict Condorcet.

¹ We have in fact proven that the Kemeny function satisfies a Duality property which is stronger than the Moderate Dual property and says: « $x \in f(X, \rho) \cap f(X, \rho)^*$ implies that x is a Borda's average».

In order to prove that f is Moderate Dual, let us suppose that candidate x is chosen for (X, p) and for $(X, p)^*$. There are three possibilities. First, x is QuasiCondorcet for (X, p) and for $(X, p)^*$. In this case $p(x, z) = p(z, x)$ for any z , implying that x is a Borda's average. Second, x is QuasiCondorcet for a situation (say (X, p)), but not for its dual. Since the Kemeny function is Inclusive Condorcet, x is chosen according to such function for the situation (X, p) and for its dual. This implies that x is a Borda's average. Third, x is QuasiCondorcet for neither (X, p) , nor $(X, p)^*$. In this case, x is chosen according to the Kemeny function for (X, p) and also for $(X, p)^*$. Therefore, x is a Borda's average.

In order to see that f Conserves Domination through Compensation it is enough to notice that, for the case in which a QuasiCondorcet candidate exists, no such candidate is dominated through compensation.

It has been proven in Pérez (1991), that for three candidates the Kemeny function coincides with the Maximin function. Therefore, it is Monotonic in the face of new voters for cases with three candidates (the case of two candidates is straightforward).

To prove that f satisfies the Choice Participation property for cases with three candidates (the case for two candidates is obvious), let $X = \{x, y, z\}$ and (X, p) be any situation. Without loss of generality, let $x \in f(X, p)$ (this means that the minimal entry in the x row of M_p is maximal) and suppose we add a voter who prefers x to y . There are three possibilities for the preferences of that voter: xyz , xzy and zxy . For the first two possibilities, $x \in f(X, p + v)$ since the Maximin SCF satisfies Weak Monotonicity in the face of new voters. Therefore, Choice Participation holds. The third possibility leads to a situation in which the entries of the y row remain unchanged while one entry of the x row is increased. In this case, the minimal entry of the x row will remain equal or higher than that of the y row, so Choice Participation also holds. \square

4. Summary and concluding remarks

Three properties have been defined with the purpose of exploring the incidence of No Show type paradoxes in the family of Condorcet Social Choice Functions: Monotonicity in the face of new voters, Weak Monotonicity in the face of new voters and Choice Participation. When one of them fails to be satisfied, a paradox occurs.

Two Duality properties—much weaker than that defined in Fishburn (1973)—and a Conservation property have also been defined, in an attempt to identify large families of Condorcet functions affected by paradoxes concerning the Monotonicity properties defined. Propositions 3, 4 and 5 show that the family of Dual functions and the family of Conserving functions are indeed affected by these paradoxes, while Proposition 6 establish that all Condorcet functions are affected by the paradox concerning the Choice Participation property. Lastly, Proposition 7 presents a function which satisfies all the

Duality and Conservation properties and which, for three or less candidates, is free from these paradoxes.

As a first conclusion concerning the Monotonicity paradoxes, in view of the intuitively weak nature of the defined Duality and Conservation properties, it seems that too many Condorcet choice functions are affected by these paradoxes.

A relevant practical question, which has not been fully studied here, refers to the number of candidates and voters that are necessary to invoke the paradoxes. Although we have not tried to find counterexamples with a minimal number of candidates and voters, the exhibited counterexamples show that 4 candidates and 22 voters, 5 and 12, 4 and 33, and 4 and 15, are enough –for Propositions 3, 4, 5 and 6, respectively– to give rise to paradoxes.

Finally, it may be interesting to analyze if the above propositions imply the occurrence of such paradoxes with the best known Condorcet choice functions, those that were defined and studied in Fishburn (1977). Let us note, without proofs, the following:

- a) The Black and Kemeny functions are not Monotonic in the face of new voters, because of Proposition 3.
- b) The Black and Fishburn functions are not Monotonic in the face of new voters, because of Proposition 4.
- c) The Nanson, Copeland and Schwartz (GOCHA) functions, and also the Black and Fishburn functions, are not Weak Monotonic in the face of new voters, because of Proposition 5.

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Resumen

Este trabajo analiza la incidencia, en el marco de las funciones de elección Condorcet, de algunas paradojas relacionadas con la llamada paradoja de abstención. Para ello se definen dos propiedades de monotonía frente a nuevos votantes y una versión para funciones de elección de la propiedad de participación de Moulin. Se demuestra que las paradojas relacionadas con la monotonía frente a nuevos votantes afectan a amplias familias de funciones de elección mientras que la paradoja relacionada con la propiedad de participación afecta a todas las funciones de elección Condorcet.

Recepción del original, octubre de 1993

Versión final, enero de 1995